GENERALIZED COUETTE FLOW WITH VARIABLE VISCOSITY AND DISSIPATION OF MECHANICAL ENERGY

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We employ the method of a small parameter to solve the problem of a pressure-stabilized Couette flow in which we take into account dissipation of mechanical energy and an exponential temperature dependence of viscosity.

We consider a stationary laminar flow of a viscous liquid in a layer between two parallel plates, y = h and y = −h, the lower plate moving in its plane with constant speed V. We assume, in addition, that in the gap between the plates a pressure gradient of constant magnitude dp/dx = A is acting. The viscosity of the liquid is assumed to vary exponentially with the temperature and the plates are kept at the constant temperatures T_1 and T_2. Then the velocity distribution and the temperature in the flow, in dimensionless variables, will be a solution of the following boundary-value problem:

$$\frac{d}{d\xi}\left(\mu\,\frac{dv}{d\xi}\right)=\varkappa, \tag{1}$$

$$\frac{d^2\theta}{d\xi^2}+\alpha\left(\frac{dv}{d\xi}\right)^2\mu=0, \tag{2}$$

$$\mu=\exp(-\theta); \tag{3}$$

for

$$\begin{aligned} &\xi=0 \quad v=0, \quad \theta=\theta_{II},\\ &\xi=1 \quad v=-1, \quad \theta=\theta_{I}. \end{aligned} \tag{4}$$

We remark that in the case of hyperbolic dependence of the viscosity on the temperature the system (1)-(2) is linear and its solution is given in [1]. In our case the system (1)-(3) is nonlinear and no exact solution for it is available. Hence, putting $\varkappa < 1$, we seek a solution of the boundary-value problem (1)-(4) by the method of a small parameter, i.e.,

$$v=v_0+\varkappa v_1+\varkappa^2 v_2+\cdots \tag{5}$$

$$\theta=\theta_0+\varkappa\theta_1+\varkappa^2\theta_2+\cdots \tag{6}$$

Expanding μ in a series in powers of $\varkappa$ and substituting Eqs. (5) and (6) into Eqs. (1) and (2), we obtain the problem for gradientless Couette flow

$$\frac{d}{d\xi}\left[\frac{dv_0}{d\xi}\exp(-\theta_0)\right]=0, \tag{7}$$

$$\frac{d^2\theta_0}{d\xi^2}+\alpha\left(\frac{dv_0}{d\xi}\right)^2\exp(-\theta_0)=0 \tag{8}$$

with the boundary conditions (4).

This problem was first solved in [2] for the special case $T_1 = T_2$; later, in [3, 4], a solution was given for arbitrary plate temperatures, the solution being of the form

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$$\theta_0 = \ln\left[C_2 \operatorname{ch}^{-2}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right)\right] \quad (C_2 > 1), \tag{9}$$

$$v_0 = \sqrt{\frac{2C_2}{\alpha}}\operatorname{th}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right) + C_4. \tag{10}$$

Consequently, for the first approximation, we have the system

$$\frac{d}{d\xi}\left[\frac{dv_1}{d\xi}\exp(-\theta_0)\right] - \left[\frac{dv_0}{d\xi}\exp(-\theta_0)\right]\frac{d\theta_1}{d\xi} = 1, \tag{11}$$

$$\frac{d^2\theta_1}{d\xi^2} + 2\alpha\frac{dv_0}{d\xi}\cdot\frac{dv_1}{d\xi}\exp(-\theta_0) - \alpha\left(\frac{dv_0}{d\xi}\right)^2\theta_1\exp(-\theta_0) = 0. \tag{12}$$

Using the solutions (9) and (10), we obtain a nonhomogeneous differential equation with variable coefficients for the temperature θ_1:

$$\frac{d^2\theta_1}{d\xi^2} + \frac{\alpha C_1^2 C_2}{\operatorname{ch}^2\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right)}\theta_1 = \frac{2C_1C_2\alpha(\xi + C_5)}{\operatorname{ch}^2\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right)}, \tag{13}$$

where C_5 is a constant of integration from Eq. (11).

It is readily established that one of the particular solutions of the homogeneous equation corresponding to Eq. (13) is

$$\theta_1^{(1)} = \operatorname{th}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right).$$

We can now readily construct the general solution of Eq. (13):

$$\theta_1 = \left[C_6 + C_7\left(\frac{C_3}{C_1}\sqrt{\frac{2}{\alpha C_2}} + \xi\right)\right]\operatorname{th}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right) - \frac{2(\xi + C_5)}{C_1} - \frac{C_7}{C_1}\sqrt{\frac{2}{\alpha C_2}}, \tag{14}$$

in accordance with which we can write Eq. (11) as

$$\frac{dv_1}{d\xi} = (\xi + C_5 + C_1\theta_1)\exp\theta_0. \tag{15}$$

Integrating Eq. (15) and taking Eq. (14) into account, we obtain

$$v_1 = C_8 - \left[\frac{1}{C_1}\sqrt{\frac{2C_2}{\alpha}}(C_5 + \xi) + \frac{C_7}{\alpha C_1}\right]\operatorname{th}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right) + \frac{2}{\alpha C_1^2}\ln\operatorname{ch}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right)$$
$$-\left[\sqrt{\frac{C_2}{2\alpha}}(C_7\xi + C_6) + \frac{C_7C_3}{C_1\alpha}\right]\operatorname{ch}^{-2}\left(C_3 + C_1\sqrt{\frac{\alpha C_2}{2}}\,\xi\right). \tag{16}$$

The integration constants C_5, C_6, C_7, and C_8 may be obtained by satisfying the zero boundary conditions providing that the system determinant

$$\Delta \neq 0. \tag{17}$$

From the hydrodynamic point of view we can look upon inequality (17) as the condition for the existence of pressure-stabilized laminar Couette flow with heat transfer and mechanical energy dissipation in the first approximation.

We note, in conclusion, that the problem of the existence of critical pressure regimes of Couette flow when the viscosity has a hyperbolic temperature dependence was considered in [5] and [6], however, in another setting.

NOTATION

v_x	is the velocity;
T, T_1, T_2	are the temperature of liquid of lower and upper plates;
V	is the velocity of lower plate;
$2h$	is the distance between plates;
μ	is the dynamic viscosity of liquid;

λ	is the liquid thermal conductivity;
J	is the mechanical equivalent of heat;
y	is the coordinate;
$v = v_x/V$	is the dimensionless velocity;
$\theta = ST$	is the dimensionless temperature;
$\varkappa = 4Ah^2/\mu_0 V$, $\alpha = \mu_0 V^2 S/J\lambda$	are the parameters;
μ_0	is the viscosity at $T = T_0 = 0$;
$\xi = h - y/2h$	is the dimensionless transverse coordinate.

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